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TIGHTNESS OF STATIONARY WAITING TIMES IN HEAVY TRAFFIC FOR $GI/GI/1$ QUEUES WITH THICK TAILS

BY

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Dedicated to the memory of Kazimierz Urbanik

Abstract. Recently, a Heavy Traffic Invariance Principle was proposed by Szczotka and Woyczyński to characterize the heavy traffic limiting distribution of normalized stationary waiting times of $G/G/1$ queues in terms of an appropriate convergence to a Lévy process. It has two important assumptions. The first of them deals with a convergence to a Lévy process of appropriate processes which is well investigated in the literature. The second one states that the sequence of appropriate normalized stationary waiting times is tight. In the present paper we characterize the tightness condition for the case of $GI/GI/1$ queues in terms of the first condition.

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1. INTRODUCTION

Consider a sequence of queueing systems of $GI/GI/1$ type with FIFO discipline of service. The n-th queueing system is generated by a sequence $\{(v_{n,k}, u_{n,k}), k \geqslant 1\}$ of pairs of nonnegative random variables $v_{n,k}$ and $u_{n,k}$ such that $\{v_{n,k}, k \geqslant 1\}$ and $\{u_{n,k}, k \geqslant 1\}$ are mutually independent and each of them is a sequence of mutually independent and identically distributed (iid) r.v.'s with distribution functions B_n and A_n, respectively, and with finite means $\bar{v}_n \stackrel{df}{=} Ev_{n,k}$ and $\bar{u}_n \stackrel{df}{=} Eu_{n,k}$ such that $a_n \stackrel{df}{=} \bar{v}_n - \bar{u}_n < 0$. Here $v_{n,k}$ represents the service time of the k-th customer in the n-th queue and $u_{n,k}$ represents the interarrival time between the arrivals of the $(k-1)$-st and k-th customers to the n-th system. Let ω_n be a stationary waiting time in the n-th queueing system, i.e.

$$\omega_n \equiv \sup_{k \geqslant 0} \sum_{j=1}^{k} (v_{n,j} - u_{n,j}).$$

It is well known that $\omega_n \xrightarrow{P} \infty$ as $a_n \uparrow 0$. The main problem of the heavy traffic theory for $GI/GI/1$ queues deals with an asymptotic of ω_n as $a_n \uparrow 0$. So the first problem is to investigate conditions on $\{(v_{n,k}, u_{n,k}), k \geqslant 1\}$ and constants c_n, $0 < c_n \uparrow \infty$, under which ω_n/c_n converge in distribution to a nondegenerated random variable W, as $a_n \uparrow 0$, i.e.

$$(1) \qquad \omega_n/c_n \xrightarrow{\mathscr{D}} W \quad \text{as } n \to \infty .$$

The second problem deals with identification of the distribution $\mathscr{L}(W)$. It is well known that if the variances $\mathrm{Var}(v_{n,1})$ and $\mathrm{Var}(u_{n,1})$ are such that $\mathrm{Var}(v_{n,1}) + \mathrm{Var}(u_{n,1})$ converge to finite positive numbers, then (1) holds with $c_n = 1/|a_n|$ and W has an exponential distribution. Boxma and Cohen [2] considered (1) in the case when the distributions $\mathscr{L}(v_{n,1})$ and $\mathscr{L}(u_{n,1})$ are Pareto distributions or Pareto distributions with some disturbances. In such situations those distributions have infinite variances. They showed that $\mathscr{L}(W)$ in (1) is the Mittag-Leffler distribution for some $\{c_n\}$ if the tail of $\mathscr{L}(v_{n,1})$ is heavier than the tail of $\mathscr{L}(u_{n,1})$, while it is the exponential distribution if the tail of $\mathscr{L}(u_{n,1})$ is heavier than the tail of $\mathscr{L}(v_{n,1})$. Boxma and Cohen [2] analyzed the problem in Laplace transform terms. Another approach to consider (1), based on a stable-Lévy approximation, was done by Whitt [11] for $GI/GI/1$ queues. A general approach to investigate (1), based on an approximation by a process with stationary increments, was done by Szczotka and Woyczyński [9], [10] for $G/G/1$ queues generated by stationary sequences $\{(v_{n,k}, u_{n,k}), k \geqslant 1\}$, which allow some dependencies between random variables. That analysis restricted to $GI/GI/1$ queues is based on an approximation by a general Lévy process and on the relation

$$\omega_n = \sup_{0 \leqslant t < \infty} \left(Z_n(t) - [nt]\,|a_n|\right), \quad \text{where } Z_n(t) = \sum_{j=1}^{[nt]} (v_{n,j} - u_{n,j} - a_n),\ t \geqslant 0,\ n \geqslant 1.$$

We analyze one of the main results from [9], formulated there as the *Heavy Traffic Invariance Principle* for queues: Assume that the following conditions hold:

I. $X_n \equiv Z_n/c_n \xrightarrow{\mathscr{D}} X$, in the Skorokhod J_1 topology in $D[0, \infty)$, with X being a Lévy process;

II. $\beta_n \equiv n\,|a_n|/c_n \to \beta$, $0 < \beta < \infty$;

III. the sequence $\{\omega_n/c_n\}$ is tight.

Then (1) holds with $W = \sup_{0 \leqslant t < \infty} (X(t) - \beta t)$.

Conditions for convergence I are well known in the literature of that subject. Namely, Prokhorov's result (formulated here in Proposition 1) states that conditions P1–P4 (given in Section 2) are necessary and sufficient for condition I to hold in the case of $GI/GI/1$ queues. Condition II deals with rates of convergences $a_n \uparrow 0$ and $c_n \uparrow \infty$. Some sufficient conditions for III are given in [9] and [10] for a general case of $G/G/1$ queues. The main aim of the paper is to characterize condition III for the case of $GI/GI/1$ queues in terms of conditions I and II or, equivalently, in terms of conditions P1–P4 and II. To describe the main results of the paper let us define

$$X_n^B(t) \stackrel{df}{=} \frac{1}{c_n}\sum_{j=1}^{[nt]}(v_{n,j}-\bar{v}_n) \quad \text{and} \quad X_n^A(t) \stackrel{df}{=} \frac{1}{c_n}\sum_{j=1}^{[nt]}(u_{n,j}-\bar{u}_n), \quad t \geqslant 0.$$

Then $X_n = X_n^B - X_n^A$. Furthermore, let G7 denote the condition that $n\bar{v}_n^2/c_n^2 \to 0$ and $n\bar{u}_n^2/c_n^2 \to 0$. The main results of the paper are formulated in Theorems 1–3. Theorem 1 states that if $X_n^B \xrightarrow{\mathscr{D}} X^B$ and $X_n^A \xrightarrow{\mathscr{D}} X^A$ in the Skorokhod J_1 topology in $D[0, \infty)$, where X^B and X^A are independent Lévy processes with $EX^B(t) = EX^A(t) = 0$ and conditions II and G7 hold, then the sequence $\{\omega_n/c_n\}$ is tight and

$$\omega_n/c_n \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t < \infty} \left(X^B(t) - X^A(t) - \beta t\right).$$

Here arises the following question. Can we characterize condition III in terms of conditions I and II? An answer to this question is positive and is given in Theorem 2: If condition I holds with $EX(t) = 0$ and furthermore conditions II, G7 and

$$\lim_{r\to\infty}\lim_{n\to\infty}\int_{-\infty}^{-r} nP(v_{n,1}-\bar{v}_n-u_{n,1}+\bar{u}_n \leqslant c_n x)\,dx = 0$$

are satisfied, then $\{\omega_n/c_n\}$ is tight and

$$\omega_n/c_n \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t < \infty} \left(X(t) - \beta t\right).$$

Another characterization of condition III, without condition G7, is given in Theorem 3 which states that if $\{B_n\}$ and $\{A_n\}$ satisfy conditions G1–G5 formulated in Section 2 and condition II holds, then $\{\omega_n/c_n\}$ is tight.

Recapitulating, the main result dealing with the conditions for $GI/GI/1$ queues under which ω_n/c_n converge is stronger than the result given in [2]. It is also stronger than an appropriate result in [10] given by case (iv) of Proposition 5 there. The main result of the paper does not assume finiteness of moments of order higher than 1 for $v_{n,1}$ and $u_{n,1}$.

The structure of the paper is as follows. In the next section we give some notation and preliminary results. The main results are formulated in Section 3, and Section 4 contains the proofs of all results.

2. PRELIMINARIES

Lévy process. Let $Y = \{Y(t), t \geqslant 0\}$ be a Lévy process (see [5]) without Gaussian component and with sample paths in the space $D[0, \infty)$. Then the *characteristic function* of $Y(t)$ has the form

$$E\exp\left(Y(iuY(t)\right) = \exp\left(t\psi_{b,\nu}(u)\right),$$

where

(2) $$\psi_{b,\nu}(u) = iub + \int_{|x|\geqslant r} (e^{iux}-1)\,\nu(dx) + \int_{0<|x|<r} (e^{iux}-1-iux)\,\nu(dx),$$

the *drift* b is a real number, the *spectral measure* ν is a positive measure on $(-\infty, \infty)$ such that $\nu(\{0\}) = 0$ and it integrates the function $\min(1, x^2)$ on $(-\infty, \infty)$, while r is a positive number such that the points $-r$ and r are continuity points of the spectral measure ν. The function $\psi_{b,\nu}(u)$ is called the *characteristic exponent* of the process Y. It is well known (see Theorem 6.1 in [5]) that $E|Y(1)|^\delta < \infty$ if and only if $\int_{|x|>1} |x|^\delta \nu(dx) < \infty$, where $\delta \geqslant 1$. In such a situation the characteristic exponent $\psi_{b,\nu}(u)$ can be written in the following form:

(3) $$\psi_{b,\nu}(u) = iub(r) + \int_{-\infty}^{\infty} (e^{iux}-1-iux)\,\nu(dx),$$

where $b(r) = b + \int_{|x|>r} x\nu(dx)$ and $b(r) = EY(1)$. Hence, if $EY(t) = 0$, then $b = -\int_{|x|>r} x\nu(dx)$.

Convergence to a Lévy process. A Lévy process can be considered as the limiting process of the processes

$$Y_n(t) = \frac{1}{c_n}\sum_{j=1}^{[nt]} \zeta_{n,j}, \qquad t \geqslant 0,\ n \geqslant 1,$$

where $\zeta_{n,j}$ are r.v.'s. In the sequel we recall some special case of the classical Prokhorov's result providing sufficient and necessary conditions for such a convergence in the case when, for each $n \geqslant 1$, $\{\zeta_{n,k},\ k \geqslant 1\}$ is a sequence of iid r.v.'s with distribution function F_n and expectation $E\zeta_{n,k} = 0$. First we introduce a definition of Prokhorov's condition for $\{F_n\}$ in which M and N are real nondecreasing functions on $(-\infty, 0)$ and $(0, \infty)$, respectively, such that $M(x) \geqslant 0$, $-N(x) \geqslant 0$ and $\lim_{x\to-\infty} M(x) = \lim_{x\to\infty} N(x) = 0$. These functions define a spectral measure ν on $(-\infty, \infty)$ by its values on the intervals (a, b) in the following way: $\nu(a, b) = M(b) - M(a)$ for $-\infty < a \leqslant b < 0$, $\nu(a, b) = N(b) - N(a)$ for $0 < a \leqslant b < \infty$ and $\nu(\{0\}) = 0$.

DEFINITION 1. A sequence of distribution functions $\{F_n\}$ satisfies the *Prokhorov condition* (shortly, *condition* P) with drift b_r and a spectral measure ν if the following conditions hold:

P1 $nF_n(yc_n) \to M(y)$ and $n(1 - F_n(xc_n)) \to -N(x)$, as $n \to \infty$, for all continuity points $y < 0$ and $x > 0$ of the functions M and N, respectively;

P2 $\lim_{x\to\infty} \sup_n n(1 - F_n(xc_n) + F_n(-xc_n)) = 0$;

P3 $b_r \stackrel{\text{df}}{=} \lim_{n\to\infty} \frac{n}{c_n} \int_{|x|\leqslant rc_n} x\,dF_n(x)$ and $|b_r| < \infty$;

P4 $\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{n}{c_n^2}\int_{|x|<\varepsilon c_n} x^2\,dF_n(x) = 0.$

PROPOSITION 1 (Prokhorov [4]). *Let, for each $n \geqslant 1$, $\{\zeta_{n,k},\ k \geqslant 1\}$ be a sequence of iid random variables with means zero, distribution functions F_n, and let Y be a Lévy process with the characteristic exponent given by* (2) *with the pair (b_r, ν). Then $Y_n \xrightarrow{\mathscr{D}} Y$ in $D[0, \infty)$ equipped with J_1 Skorokhod topology if and only if $\{F_n\}$ satisfies condition* P *with drift b_r and a spectral measure ν.*

In the paper we consider a situation when a Lévy process Y from Proposition 1 satisfies $EY(t) = 0$, which is equivalent to the following condition:

P5 $\int_{|x|>1} |x|\,\nu(dx) < \infty$ and $b_r = -\int_{|x|>r} x\nu(dx).$

But $\int_{|x|>r} x\nu(dx) = \int_{-\infty}^{-r} x dM(x) + \int_r^{\infty} x dN(x)$. Therefore, using the formula for integrating by parts, we see that under condition P5 the following holds:

$$b_r = rM(-r) + rN(r) + \int_{-\infty}^{-r} M(x)\,dx + \int_r^{\infty} N(x)\,dx. \tag{4}$$

Let $F_n^B(x) = P(v_{n,k} - \bar{v}_n \leqslant x)$ and $F_n^A(x) = P(u_{n,k} - \bar{u}_n \leqslant x)$. Then immediately from Proposition 1 we infer that if $\{F_n^B\}$ and $\{F_n^A\}$ satisfy conditions P1–P4 with pairs (b_r^B, ν^B) and (b_r^A, ν^A), respectively, then

$$X_n \equiv X_n^B - X_n^A \xrightarrow{\mathscr{D}} X^B - X^A,$$

where X^B and X^A are Lévy processes with pairs (b_r^B, ν^B) and (b_r^A, ν^A), respectively.

In the paper we use also a reverse result to the above in some sense. Namely, let

$$\zeta_{n,k} \stackrel{df}{=} (v_{n,k} - \bar{v}_n) - (u_{n,k} - \bar{u}_n),\ n, k \geqslant 1, \quad \text{and} \quad F_n^{B,A}(x) = P(\zeta_{n,k} \leqslant x).$$

PROPOSITION 2 (see [8], Theorem 1). *Let $\{F_n^{B,A}\}$ satisfy conditions* P1–P4 *with functions N, M, the pair (b_r, ν) and assume that the following conditions hold*:

$$\frac{\bar{v}_n}{c_n} \to 0, \quad \frac{\bar{u}_n}{c_n} \to 0, \tag{5}$$

$$\lim_{n\to\infty} \int_{-\infty}^{-r} nF_n^{B,A}(xc_n)\,dx = \int_{-\infty}^{-r} M(x)\,dx. \tag{6}$$

Then $\{F_n^B\}$ and $\{F_n^A\}$ satisfy conditions P1–P4 *with pairs (b_r^B, ν^B) and (b_r^A, ν^A), respectively, where $b_r = b_r^B - b_r^A$, $\nu^B(a, b) = \nu(a, b)$, $\nu^A(a, b) = \nu(-b, -a)$ for $0 < a \leqslant b$ and $\nu^B(a, b) = \nu^A(a, b) = 0$ for $a \leqslant b < 0$.*

Modifications of conditions P1–P4. Conditions P1–P4 deal with centered r.v.'s by expected values. Here we consider some their modifications for not centered positive r.v.'s. In Proposition 3 we show that those two sets of conditions are equivalent under some additional assumptions. The conditions introduced here are more natural for our analysis of tightness of $\{\omega_n/c_n\}$.

Let $\{\eta_{n,k},\ k \geqslant 1,\ n \geqslant 1\}$ be an array of nonnegative r.v.'s which are iid for each $n \geqslant 1$ with distribution functions G_n and finite means $\kappa_n \stackrel{df}{=} E\eta_{n,k}$. Introduce the following notation for conditions on $\{G_n\}$:

G1 $\lim\limits_{n\to\infty} n(1-G_n(xc_n)) = -\tilde{N}^G(x)$ for the continuity points $x>0$ of $\tilde{N}^G$;

G2 $\lim\limits_{x\to\infty} \sup\limits_n n(1-G_n(xc_n)) = 0$;

G3 $\tilde{b}_r^G \stackrel{df}{=} -\lim\limits_{n\to\infty} \frac{n}{c_n}\int_{rc_n}^{\infty} x\,dG_n(x)$ and $|\tilde{b}_r^G| < \infty$ for some $0<r<\infty$;

G4 $\lim\limits_{\varepsilon\to 0} \limsup\limits_{n\to\infty} \frac{n}{c_n^2}\int_0^{\varepsilon c_n} x^2\,dG_n(x) = 0$;

G5 $\int_1^{\infty} x\tilde{\nu}^G(dx) < \infty$ and $\tilde{b}_r^G = -\int_r^{\infty} x\tilde{\nu}^G(dx)$, where $\tilde{\nu}^G$ is defined by $\tilde{N}^G$;

G6 $\lim\limits_{n\to\infty} \kappa_n/c_n = 0$;

G7 $\lim\limits_{n\to\infty} n\kappa_n^2/c_n^2 = 0$.

Notice that G7 implies G6 and it specifies the rate of convergence G6.

Define $F_n^G(x) \stackrel{df}{=} P(\eta_{n,k}-\kappa_n \leqslant x) \equiv G_n(x+\kappa_n)$ for $-\infty < x < \infty$ and $N^G \stackrel{df}{=} N$, $M^G \stackrel{df}{=} M$, $\nu^G \stackrel{df}{=} \nu$ in condition P1 for $\{F_n^G\}$.

From the relation $F_n^G(xc_n) = G_n(c_n(x+\kappa_n/c_n))$ it follows that under condition G6 we obtain $M^G(y) = \lim_{n\to\infty} F_n^G(yc_n) = 0$, which in turn implies that the spectral measure ν^G has support in $(0,\infty)$. The following proposition gives some relations between conditions P1–P4 and G1–G4.

Proposition 3. *If condition* G6 *holds, then the following relations are satisfied*:

(i) $\{F_n^G\}$ *satisfies* P1 *with* $N^G = \tilde{N}^G$ *iff* $\{G_n\}$ *satisfies* G1 *with* $\tilde{N}^G = N^G$.

(ii) $\{F_n^G\}$ *satisfies* P2 *iff* $\{G_n\}$ *satisfies* G2.

(iii) *If* G1 *holds, then* $\{F_n^G\}$ *satisfies* P3 *with* $b_r = \tilde{b}_r^G$ *iff* $\{G_n\}$ *satisfies* G3.

(iv) *If* G3 *holds and* $\{G_n\}$ *satisfies* G4, *then* $\{F_n^G\}$ *satisfies* P4.

Reverse: *if* G1, G3 *and* G7 *hold and* $\{F_n^G\}$ *satisfies* P4, *then* $\{G_n\}$ *satisfies* G4.

(v) *Conditions* P5 *for* $\{F_n^G\}$ *and* G5 *are equivalent with* $\tilde{b}_r^G = b_r^G$.

From Proposition 3 we obtain the following remark.

Remark 1. Under the condition G7 the set of conditions P1–P4 is equivalent to the set of conditions G1–G4.

3. MAIN RESULTS

The main results of the paper are given in the following two theorems.

THEOREM 1. *Let* $X_n^B \xrightarrow{\mathscr{D}} X^B$ *and* $X_n^A \xrightarrow{\mathscr{D}} X^A$ *in the Skorokhod* J_1 *topology in* $D[0, \infty)$, *where* X^B *and* X^A *are independent Lévy processes with* $EX^B(t) = EX^A(t) = 0$ *and let conditions* II *and* G7 *hold. Then the sequence* $\{\omega_n/c_n\}$ *is tight and*

$$\omega_n/c_n \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t < \infty} (X^B(t) - X^A(t) - \beta t).$$

THEOREM 2. *Let condition* I *hold with* $EX(t) = 0$ *and assume that conditions* II *and* G7 *are satisfied. Furthermore, let*

$$\lim_{r \to \infty} \lim_{n \to \infty} \int_{-\infty}^{-r} nF_n^{B,A}(xc_n)\, dx = 0. \tag{7}$$

Then $\{\omega_n/c_n\}$ *is tight and*

$$\omega_n/c_n \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t < \infty} (X(t) - \beta t).$$

THEOREM 3. *Let* $\{B_n\}$ *and* $\{A_n\}$ *satisfy conditions* G1–G5 *with spectral Lévy measures* ν^A *and* ν^B, *respectively, and let condition* II *hold. Then the sequence* $\{\omega_n/c_n\}$ *is tight.*

The proofs of Theorems 1 and 2 are based on Propositions 1–3 from Section 2 and on Lemmas 1–3 formulated below. The proof of Theorem 3 is based on Lemmas 1–3. In those lemmas we use the notation ω_n^B and ω_n^A for the stationary waiting times in some $M/GI/1$ and $GI/M/1$ queueing systems, respectively. To define them let us denote by $\{y_{n,k},\ k \geqslant 1\}$ a sequence of iid r.v.'s, mutually independent of the sequences $\{v_{n,k},\ k \geqslant 1\}$ and $\{u_{n,k},\ k \geqslant 1\}$ and such that $y_{n,k}$ are exponentially distributed with mean

$$\bar{y}_n \equiv Ey_{n,k} \stackrel{df}{=} \bar{v}_n + \tfrac{1}{2}(\bar{u}_n - \bar{v}_n).$$

Since $\bar{v}_n - \bar{y}_n = a_n/2 < 0$ and $\bar{y}_n - \bar{u}_n = a_n/2 < 0$, the quantities

$$\omega_n^B \stackrel{df}{=} \sup_{0 \leqslant k < \infty} \sum_{j=1}^{k} (v_{n,j} - y_{n,j}) \quad \text{and} \quad \omega_n^A \stackrel{df}{=} \sup_{0 \leqslant k < \infty} \sum_{j=1}^{k} (y_{n,j} - u_{n,j}) \tag{8}$$

are finite with probability one. Hence we get the following inequality:

$$\omega_n \leqslant \omega_n^B + \omega_n^A. \tag{9}$$

LEMMA 1. *If the sequences* $\{\omega_n^B/c_n\}$ *and* $\{\omega_n^A/c_n\}$ *are tight, then* $\{\omega_n/c_n\}$ *is tight.*

LEMMA 2. *Let $\{B_n\}$ satisfy conditions* G1–G4 *and let condition* II *hold. Then for each $s \geq 0$ the following convergence holds*:

$$\lim_{n\to\infty} E\big(\exp(-s\omega_n^B/c_n)\big) = \left(1+\frac{2}{s\beta}\psi^B(s)\right)^{-1} \equiv \Psi^B(s), \tag{10}$$

where

$$\psi^B(s) = -sb_r^B + \int_r^\infty (e^{-sx}-1)\,\nu^B(dx) + \int_0^r (e^{-sx}-1+sx)\,\nu^B(dx) \tag{11}$$

and ν^B is the spectral measure defined by $\tilde{N}^B$ from G1. *Moreover, if $\{B_n\}$ satisfies condition* G5, *then*

$$\psi^B(s) = \int_0^\infty (e^{-sx}-1+sx)\,\nu^B(dx)$$

and $\Psi^B(s)$ is the Laplace–Stieltjes transform of a probability measure on $[0, \infty)$.

Let $\lambda_{0,n}$ be a number from the interval (0, 1), being the root of the following equation:

$$s - \hat{A}_n\left(\frac{1-s}{\bar{v}_n}\right) = 0, \tag{12}$$

where $\hat{A}_n$ denotes the Laplace–Stieltjes transform of A_n.

LEMMA 3. *Let $\{A_n\}$ satisfy conditions* G1–G5 *and condition* II. *Then*

$$\liminf_n \frac{(1-\lambda_{0,n})\,c_n}{\bar{v}_n} > 0 \tag{13}$$

and the sequence $\{\omega_n^A/c_n\}$ is tight. Moreover, if

$$0 < \lambda \stackrel{df}{=} \lim_n \frac{c_n(1-\lambda_{0,n})}{\bar{v}_n} < \infty, \tag{14}$$

then

$$P(\omega_n^A/c_n > x) \to e^{-\lambda x}, \qquad \text{for } x \geq 0, \text{ as } n\to\infty. \tag{15}$$

From the proofs of Lemmas 2 and 3 we get immediately the following corollary.

COROLLARY 1. *If each subsequence of the sequences $\{A_n\}$ and $\{B_n\}$ satisfies conditions* G1–G5, *then $\{\omega_n/c_n\}$ is tight.*

4. PROOFS

In this section we give proofs of Proposition 3, Lemmas 1–3 and Theorems 1 and 2.

Proof of Proposition 3

(i) First notice that in view of $F_n^G(yc_n) = G_n(c_n(y+\kappa_n/c_n))$ and G6 we get

$$M^G(y) = \lim_n F_n^G(yc_n) = \lim_n G_n(c_n(y+\kappa_n/c_n)) = 0 \quad \text{for } y < 0.$$

Now observe that the following relations hold:

$$G_n(c_n x) = F_n^G(c_n x - \kappa_n) \leqslant F_n^G(c_n x) = G_n(c_n(x+\kappa_n/c_n)),$$
$$F_n^G(c_n x) = G_n(c_n x + \kappa_n) \geqslant G_n(c_n x) = F_n^G(c_n(x-\kappa_n/c_n)).$$

For $\varepsilon > 0$ let n_0 be such that for $n \geqslant n_0$ the inequality $\kappa_n/c_n \leqslant \varepsilon$ holds, which is guaranteed by G6. Consequently, we obtain

$$1-G_n(c_n x) \geqslant 1-F_n^G(c_n x) \geqslant 1-G_n(c_n(x+\varepsilon)),$$
$$1-F_n^G(c_n x) \leqslant 1-G_n(c_n x) \leqslant 1-F_n^G(c_n(x-\varepsilon)).$$

Hence, if x is a continuity point of ν^G, then the above implies

$$\lim_n n(1-G_n(c_n x)) = \lim_n (1-F_n^G(c_n x)),$$

which proves equivalence (i).

(ii) The proof of equivalence (ii) runs over in a similar way to the proof of equivalence (i).

The proofs of (iii) and (iv) are given here in terms of r.v.'s $\eta_n \equiv \eta_{n,1}$ with distribution functions G_n and means κ_n, respectively. Then F_n^G are distribution functions of $\eta_n - \kappa_n$, respectively, and $I(A)$ denotes the indicator of a set A.

(iii) Notice that

$$(16) \quad E(\eta_n-\kappa_n)I(|\eta_n-\kappa_n| \leqslant rc_n) = -E(\eta_n-\kappa_n)I(|\eta_n-\kappa_n| > rc_n)$$
$$= -E\eta_n I(\eta_n > rc_n) + E\eta_n I(rc_n < \eta_n \leqslant rc_n+\kappa_n)$$
$$-E\eta_n I(\eta_n < -rc_n) - E\eta_n I(-rc_n < \eta_n \leqslant -rc_n+\kappa_n) + \kappa_n P(|\eta_n-\kappa_n| > rc_n).$$

If G1 and G6 hold, then by equivalence (i) we get

$$(17) \quad \lim_{n\to\infty} \frac{n}{c_n} E\eta_n I(rc_n < \eta_n \leqslant rc_n+\kappa_n)$$
$$\leqslant \lim_{n\to\infty} (r+\kappa_n/c_n) nP(rc_n < \eta_n \leqslant c_n(r+\kappa_n/c_n)) = 0.$$

In a similar way we obtain

$$(18) \qquad \lim_{n\to\infty} \frac{n}{c_n} E\eta_n I(-rc_n < \eta_n \leqslant -c_n(r-\kappa_n/c_n)) = 0.$$

Hence, if G1 and G6 hold, then by equivalence (i) and next by (17), (18) and (16) we get

$$(19) \quad \lim_{n\to\infty} \frac{n}{c_n} E(\eta_n-\kappa_n)I(|\eta_n-\kappa_n| \leqslant rc_n) = -\lim_{n\to\infty} \frac{n}{c_n} E\eta_n I(\eta_n > rc_n),$$

which proves (iii).

(iv) Let $\varepsilon > 0$ and n_0 be such that for $n \geqslant n_0$ the inequality $\kappa_n/c_n \leqslant \varepsilon$ holds. To prove that G4 implies P4 under G1 and G6 notice that for $n \geqslant n_0$ the following inclusions of sets hold:

$$\{|\eta_n - \kappa_n| \leqslant \varepsilon c_n\} \subseteq \{\eta_n \leqslant \varepsilon c_n + \kappa_n\} \subseteq \{\eta_n \leqslant 2\varepsilon c_n\}.$$

Hence for $n \geqslant n_0$ we have

$$E\left(\frac{\eta_n - \kappa_n}{c_n}\right)^2 I\left(\frac{|\eta_n - \kappa_n|}{c_n} \leqslant \varepsilon\right) \leqslant E\left(\frac{\eta_n}{c_n}\right)^2 I\left(\frac{\eta_n}{c_n} \leqslant 2\varepsilon\right) + 2\frac{\kappa_n}{c_n} E\frac{\eta_n}{c_n} I\left(\frac{\eta_n}{c_n} > 2\varepsilon\right),$$

which shows that G4 implies P4.

To prove that P4 implies G4 under G1 and G7 notice that

$$\left\{\frac{\eta_n}{c_n} < \varepsilon\right\} \subseteq \left\{\frac{\eta_n - \kappa_n}{c_n} < \varepsilon\right\} = \left\{-\varepsilon < \frac{\eta_n - \kappa_n}{c_n} < \varepsilon\right\} \cup \left\{\frac{\eta_n}{c_n} < \frac{\kappa_n}{c_n} - \varepsilon\right\}.$$

Hence for $\kappa_n/c_n < \varepsilon$ we have

$$\left\{\frac{\eta_n}{c_n} < \varepsilon\right\} \subseteq \left\{\frac{|\eta_n - \kappa_n|}{c_n} < \varepsilon\right\} \equiv A(n).$$

Therefore

$$nE\left(\frac{\eta_n}{c_n}\right)^2 I\left(\frac{\eta_n}{c_n} < \varepsilon\right) \leqslant nE\left(\frac{\eta_n}{c_n}\right)^2 I(A(n)) = nE\left(\frac{\eta_n - \kappa_n}{c_n} + \frac{\kappa_n}{c_n}\right)^2 I(A(n))$$

$$= nE\left(\frac{\eta_n - \kappa_n}{c_n}\right)^2 I(A(n)) + 2\frac{\kappa_n}{c_n} nE\left(\frac{\eta_n - \kappa_n}{c_n}\right) I(A(n)) + \left(\frac{\kappa_n}{c_n}\right)^2 nP(A(n)).$$

Denote by $D_{n,i}$, $i = 1, 2, 3$, the i-th component of the above sum. Applying condition P4 to component $D_{n,1}$, we get

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} D_{n,1} = 0.$$

Next, applying conditions G1, G3 and G6, which imply P3, to component $D_{n,2}$, we obtain

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} D_{n,2} = 0.$$

Finally, applying condition G7 to component $D_{n,3}$, we have

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} D_{n,3} = 0.$$

Hence by the inequality

$$\frac{n}{c_n^2} E\eta_n^2 I(\eta_n < \varepsilon c_n) \leqslant D_{n,1} + D_{n,2} + D_{n,3}$$

we get

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{n}{c_n^2}E\eta_n^2 I(\eta_n<\varepsilon c_n)=0.$$

This ends the proof of (iv).

The proof of the equivalence of P5 and G5 is obvious. This completes the proof of Proposition 3. ∎

Proof of Lemma 1. The proof of Lemma 1 follows from inequality (9). ∎

In the sequel we use the notation $\hat{B}_n$ and $\hat{A}_n$ for the Laplace–Stieltjes transforms of $\{B_n\}$ and $\{A_n\}$, respectively.

Proof of Lemma 2. Using the form of the distribution function for the stationary waiting time in $M/GI/1$ queue, given in [3], p. 255, formula (4.82), and next applying the consideration from Section 4.2 in [9], we get the following form of the Laplace–Stieltjes transform for ω_n^B/c_n:

$$E\exp(-s\omega_n^B/c_n)=\left(1+\frac{1}{s\beta_n^B}n\int_0^\infty(\exp(-sx/c_n)-1+sx/c_n)\,dB_n(x)\right)^{-1}, \tag{20}$$

where

$$\beta_n^B\equiv\frac{n|\bar{v}_n-\bar{y}_n|}{c_n}=\frac{n|a_n|}{2c_n}=\frac{\beta_n}{2}$$

and $\beta_n^B\to\beta/2$ by condition II. Now by the definition of $\hat{B}_n(s)$ we obtain

$$n\big(\hat{B}_n(s/c_n)-1+s\bar{v}_n/c_n\big)$$
$$=n\int_0^\infty(\exp(-sx/c_n)-1+sx/c_n)\,dB_n(x)=n\int_0^\infty(e^{-sx}-1+sx)\,dB_n(xc_n).$$

But

$$\int_0^\infty(e^{-sx}-1+sx)\,dB_n(xc_n)=\int_0^\varepsilon(e^{-sx}-1+sx)\,dB_n(xc_n) \tag{21}$$
$$+\int_\varepsilon^r(e^{-sx}-1+sx)\,dB_n(xc_n)+s\int_r^\infty x\,dB_n(xc_n)+\int_r^\infty(e^{-sx}-1)\,dB_n(xc_n)$$
$$\equiv C_{n,1}+C_{n,2}+C_{n,3}+C_{n,4},$$

where ε and r are continuity points of the spectral measure ν. Using the inequality $e^{-sx}-1+sx\leqslant s^2x^2$ and condition G4 to the first component on the right-hand side of the above equality, i.e. to $nC_{n,1}$, we get

$$\lim_{\varepsilon\to\infty}\limsup_{n\to\infty}n\int_0^\varepsilon(e^{-sx}-1+sx)\,dB_n(xc_n)\leqslant\lim_{\varepsilon\to\infty}\limsup_{n}n\int_0^\varepsilon(sx)^2\,dB_n(xc_n) \tag{22}$$
$$=\lim_{\varepsilon\to 0}\limsup_{n}\frac{n}{c_n^2}s^2\int_0^{c_n\varepsilon}x^2\,dB_n(x)=0.$$

Applying condition G3 to the third component of (21), i.e. to $nC_{n,3}$, we obtain

$$sn\int_r^\infty x\,dB_n(xc_n) = s\frac{n}{c_n}\int_{rc_n}^\infty x\,dB_n(x) \to -sb_r^B. \tag{23}$$

To consider components $nC_{n,2}$ and $nC_{n,4}$ let us define measures ν_n^B on $(0, \infty)$ by $\nu_n^B(a, b) = n\big(B_n(bc_n)-B_n(ac_n)\big)$ for $0 < a \leqslant b$. Then by G1 and G2 we have $\nu_n(a, b) \to \nu^B(a, b) \equiv \tilde{N}^B(b)-\tilde{N}^B(a)$, as $n \to \infty$, if a and b are continuity points of the spectral measure ν^B. Furthermore, let us define probability measures $\tilde{\nu}_n$ and $\tilde{\nu}$ on $[\varepsilon, \infty)$ by $\tilde{\nu}_n = \nu_n/\nu_n(\varepsilon, \infty)$ and $\tilde{\nu} = \nu^B/\nu^B(\varepsilon, \infty)$, respectively. Since ε is a continuity point of ν^B, by conditions G1 and G2 we get the weak convergence $\tilde{\nu}_n \Rightarrow \tilde{\nu}$. The functions $e^{-sx}-1$ and $e^{-sx}-1+sx$ of variable x are continuous on $[0, \infty)$. Furthermore, they are bounded on $[\varepsilon, \infty)$ and $[\varepsilon, r]$, respectively. These facts jointly with $\tilde{\nu}_n \Rightarrow \tilde{\nu}$ give the following convergences:

$$n\int_\varepsilon^r (e^{-sx}-1+sx)\,dB_n(xc_n) = \int_\varepsilon^r (e^{-sx}-1+sx)\,\nu_n^B(dx) \to \int_\varepsilon^r (e^{-sx}-1+sx)\,\nu^B(dx)$$

and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} n\int_\varepsilon^r (e^{-sx}-1+sx)\,dB_n(xc_n) = \int_0^r (e^{-sx}-1+sx)\,\nu^B(dx). \tag{24}$$

Similarly we get

$$n\int_r^\infty (e^{-sx}-1)\,dB_n(xc_n) = \int_r^\infty (e^{-sx}-1)\,\nu_n^B(dx) \to \int_r^\infty (e^{-sx}-1)\,\nu^B(dx). \tag{25}$$

Now compiling convergences (22)–(25) with (21) we get the first assertion of Lemma 2, i.e. (10) and (11).

To prove the second assertion of the lemma we need to show that $\Psi^B(s)$ is a continuous function at $s = 0$, i.e. $\lim_{s\to 0}\Psi^B(s) = 1$. This holds if $\lim_{s\to 0}\psi^B(s)/s = 0$. But

$$\lim_{s\to 0}\frac{\psi^B(s)}{s} = -b_r^B - \int_r^\infty x\nu^B(dx).$$

Hence by assumption G5 we get $\lim_{s\to 0}\psi^B(s)/s = 0$. This completes the proof of the lemma. ∎

Proof of Lemma 3. Since $\lambda_{0,n}$ satisfies equation (12), we have

$$\lambda_{0,n} - \hat{A}_n\left(\frac{1-\lambda_{0,n}}{\bar{v}_n}\right) = 0. \tag{26}$$

Putting

$$z_n \stackrel{df}{=} c_n\frac{1-\lambda_{0,n}}{\bar{v}_n},$$

we get $\lambda_{0,n} = 1-z_n\bar{v}_n/c_n$, which together with (26) give the equality

$$1-z_n\frac{\bar{v}_n}{c_n}-\hat{A}_n\left(\frac{z_n}{c_n}\right) = 0.$$

Hence

$$1+z_n\frac{|a_n|}{c_n} = \hat{A}_n\left(\frac{z_n}{c_n}\right)+z_n\frac{\bar{u}_n}{c_n},$$

which in turn gives

$$z_n\beta_n = n\left(\hat{A}_n\left(\frac{z_n}{c_n}\right)-1+\frac{z_n}{c_n}\bar{u}_n\right). \tag{27}$$

But

$$n(\hat{A}_n(z_n/c_n)-1+\bar{u}_n z_n/c_n) = n\int_0^{\infty}(\exp(-xz_n/c_n)-1+xz_n/c_n)\,dA_n(x)$$

$$= n\int_0^{rc_n}(\exp(-xz_n/c_n)-1+xz_n/c_n)\,dA_n(x)+n\int_{rc_n}^{\infty}(\exp(-xz_n/c_n)-1+xz_n/c_n)\,dA_n(x)$$

$$\leqslant nz_n^2\frac{1}{c_n^2}\int_0^{rc_n}x^2\,dA_n(x)+n\int_{rc_n}^{\infty}\left(1+\frac{z_n x}{c_n}-1+\frac{z_n x}{c_n}\right)dA_n(x)$$

$$\leqslant nz_n^2\frac{1}{c_n^2}\int_0^{rc_n}x^2\,dA_n(x)+2\frac{nz_n}{c_n}\int_{rc_n}^{\infty}x\,dA_n(x).$$

Consequently, by (27), we get

$$\beta_n \leqslant z_n n\frac{1}{c_n^2}\int_0^{rc_n}x^2\,dA_n(x)+2\frac{n}{c_n}\int_{rc_n}^{\infty}x\,dA_n(x). \tag{28}$$

But by conditions G4 and G3 we have

$$\limsup_{n\to\infty} n\frac{1}{c_n^2}\int_0^{rc_n}x^2\,dA_n(x) < \infty \quad \text{and} \quad \lim_n \frac{n}{c_n}\int_{rc_n}^{\infty}x\,dA_n(x) = -b_r^A, \tag{29}$$

respectively. Therefore, if $z_n \to 0$, then in view of $\lim_n \beta_n = \beta$ and next in virtue of (28) and (29), and finally by $\lim_{r\to\infty} b_r^A = 0$ we obtain $\beta \leqslant 0$, which contradicts the assumption that $0 < \beta < \infty$. Hence

$$\liminf_{n\to\infty}(1-\lambda_{0,n})\,c_n/\bar{v}_n > 0.$$

It is well known that for $GI/M/1$ queues the distribution function of ω_n has the form

$$P(\omega_n > x) = \lambda_{0,n}\exp(-(1-\lambda_{0,n})\,x/\bar{v}_n) \quad \text{for } x \geqslant 0$$

(see [3], p. 230, equation (2.98)).

Hence

$$P(\omega_n > xc_n) = \lambda_{0,n}\exp\left(-(1-\lambda_{0,n})c_n x/\bar{v}_n\right) \quad \text{for } x \geqslant 0$$

and

$$\limsup_{n\to\infty} P(\omega_n/c_n > x) = \limsup_{n\to\infty} \lambda_{0,n}\exp\left(-(1-\lambda_{0,n})c_n x/\bar{v}_n\right) \tag{30}$$

$$\leqslant \limsup_{n\to\infty} \exp\left(-(1-\lambda_{0,n})c_n x/\bar{v}_n\right).$$

The above jointly with assertion (13) imply that the sequence $\{\omega_n/c_n\}$ is tight both in the case when $0 < \liminf_{n\to\infty}(1-\lambda_{0,n})c_n/\bar{v}_n < \infty$ as well as in the case when $\limsup_{n\to\infty}(1-\lambda_{0,n})c_n/\bar{v}_n = \infty$.

To prove the second assertion of the lemma notice that under (14) we have $\lambda_{0,n} \to 1$, which by the second equality in formula (30) gives the second assertion of the lemma. This completes the proof of the lemma. ■

Proof of Theorem 1. By the assumptions $X_n^B \xrightarrow{\mathscr{D}} X^B$, $X_n^A \xrightarrow{\mathscr{D}} X^A$ and $EX^B(t) = EX^A(t) = 0$ and by Proposition 1 it follows that the sequences $\{F_n^B\}$ and $\{F_n^A\}$ satisfy conditions P1–P5. This in turn and condition G7 imply that the sequences $\{B_n\}$ and $\{A_n\}$ satisfy conditions G1–G7. Now by Lemma 2 we see that $\{\omega_n^B/c_n\}$ is tight and, by Lemma 3, $\{\omega_n^A/c_n\}$ is tight. This in turn and Lemma 1 imply that $\{\omega_n/c_n\}$ is tight. Hence conditions I, II and III hold, which implies that

$$\omega_n/c_n \xrightarrow{\mathscr{D}} \sup_{0\leqslant t<\infty} \left(X^B(t) - X^A(t) - \beta t\right).$$

This completes the proof of the theorem. ■

Proof of Theorem 2. By the assumptions of Theorem 2 and by Proposition 1 it follows that $\{F_n^{B,A}\}$ satisfies conditions P1–P5. This together with assumption (7) imply by Proposition 2 that $\{F_n^B\}$ and $\{F_n^A\}$ satisfy conditions P1–P5. That in turn and condition G7 imply that $\{B_n\}$ and $\{A_n\}$ satisfy conditions G1–G7. Therefore, from Lemmas 2 and 3 we infer that $\{\omega_n^B/c_n\}$ and $\{\omega_n^A/c_n\}$ are tight, which by Lemma 1 implies that $\{\omega_n/c_n\}$ is tight. Hence conditions I, II and III are satisfied, which implies the convergence

$$\omega_n/c_n \xrightarrow{\mathscr{D}} \sup_{0\leqslant t<\infty} \left(X(t) - \beta t\right).$$

This completes the proof of the theorem. ■

Proof of Theorem 3. The proof of the theorem follows immediately from Lemmas 1–3. ■

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